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Complete Monotonicity and Reduction Formulas for Certain Kampé de Fériet Functions

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Abstract

We extend the classical Euler-type integral representations for the Appell functions F_1, F_2, and F_3, to the appropriate Kampé de Fériet functions by using integration against the Meijer–Nørlund G-function. In particular, these representations provide analytic continuation of the corresponding Kampé de Fériet functions. We further focus on the following two applications. First, we obtain sufficient conditions for complete monotonicity on the positive quadrant for three families of the Kampé de Fériet functions. These conditions can be expressed directly in terms of parameters and imply, among other things, joint log-convexity and related inequalities for partial derivatives of the Kampé de Fériet functions. Second, we show how known reduction and transformation formulas for the Appell and the generalized hypergeometric functions can be lifted to Kampé de Fériet functions by concatenating parameter arrays via the integral representations. This yields several reduction formulas, including extensions of some classical and new product identities. Further combining integration against the Meijer–Nørlund G-function with Slater's double series transformation we obtain several exotic identities for infinite sums of the generalized hypergeometric functions.

Keywords: Kampé de Fériet function; generalized hypergeometric function; Meijer–Nørlund G function; complete monotonicity; reduction formula; integral representation

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1. Introduction and Preliminaries

The realm of bivariate hypergeometric series is much richer than that of univariate hypergeometric series. Following the classical works of Horn published over the period of around fifty years, see, for example [1,2], a formal power series in two variables,

$$\sum_{m,n=0}^{\infty} c_{m,n} x^m y^n, \quad (x,y) \in \mathbb{C}^2,$$

is called hypergeometric if the functions

$$f(m,n) := \frac{c_{m+1,n}}{c_{m,n}}, \quad g(m,n) := \frac{c_{m,n+1}}{c_{m,n}}$$

are both rational [3] (Section 5.7). The literature on multivariate hypergeometric functions is enormous and, as such, we will not make any survey in this short note. An interested

reader is referred to the books by Karlsson and Srivastava [4], and Exton [5], or more recent brief surveys by Schlosser [6] and Matsumoto [7]. A detailed account of convergence and classification of this type of series can be found in the recent paper [8]. Integral representations for bivariate hypergeometric functions have been considered, for instance, in [9–13]. Reduction formulas for the Kampé de Fériet function have been explored, for example, in [14–16]. More references will be given throughout the paper in connection with specific facts and formulas.

Before proceeding, let us introduce some notation. We will use the standard Pochhammer symbol $(a)_n = \Gamma(a+n)/\Gamma(a)$ and shorthand notation $(\mathbf{a})_n = (a_1)_n (a_2)_n \cdots (a_p)_n$ for products over the components of a vector $\mathbf{a} = (a_1, \ldots, a_p)$. Expressions like $\mathbf{a} + \alpha$ and $\Re(\mathbf{a}) > 0$, where α is a scalar, will be understood component-wise. One particularly useful class of hypergeometric functions of two variables was introduced by Kampé de Fériet in [17] and is defined by (using the modified notation from [4] Equation (28))

$$F_{q:r;s}^{p:k;l}\left[\begin{matrix}\mathbf{a}:\mathbf{b};\mathbf{c}\\ \mathbf{d}:\mathbf{e};\mathbf{f}\end{matrix};x,y\right] = \sum_{m,n=0}^{\infty} \frac{(\mathbf{a})_{m+n}\,(\mathbf{b})_m (\mathbf{c})_n}{(\mathbf{d})_{m+n}\,(\mathbf{e})_m\,(\mathbf{f})_n} \frac{x^m}{m!}\,\frac{y^n}{n!}, \tag{1}$$

where $\mathbf{a} \in \mathbb{C}^p$, $\mathbf{b} \in \mathbb{C}^k$, $\mathbf{c} \in \mathbb{C}^l$, $\mathbf{d} \in \mathbb{C}^q$, $\mathbf{e} \in \mathbb{C}^r$, $\mathbf{f} \in \mathbb{C}^s$. The double series (1) converges if one of the following is true [4] (Equations (28) and (29)):

(i) $p + k < q + r + 1$, $p + l < q + s + 1$ for all $|x|, |y| < \infty$;
(ii) $p + k = q + r + 1$, $p + l = q + s + 1$ for $|x|, |y| < 1$, when $p \le q$, and for $|x|^\alpha + |y|^\alpha < 1$, $\alpha = 1/(p - q)$, when $p > q$.

See the recent papers [8] for further details on convergence, and [13] for various other aspects of the Kampé de Fériet function.

A product of univariate generalized hypergeometric functions is a special case of the Kampé de Fériet function:

$${}_pF_q\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| x\right) {}_kF_l\left(\begin{matrix}\mathbf{c}\\ \mathbf{d}\end{matrix}\middle| y\right) = F_{0:q;l}^{0:p;k}\left[\begin{matrix}-:\mathbf{a};\mathbf{c}\\ -:\mathbf{b};\mathbf{d}\end{matrix};x,y\right], \tag{2}$$

where, as usual, [18] Equation (2.1.2), [19] Equation (16.2 (i))

$${}_pF_q\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| z\right) = {}_pF_q(\mathbf{a};\mathbf{b};z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n n!} z^n = \sum_{n=0}^{\infty} \frac{(\mathbf{a})_n}{(\mathbf{b})_n n!} z^n.$$

This series converges for all $z \in \mathbb{C}$ if $p \le q$ and for $|z| < 1$ if $p = q + 1$. In the latter case, the function ${}_pF_q(z)$ is defined in $\mathbb{C} \setminus [1, \infty)$ by analytic continuation. Appell's functions F_i, $i = 1, \ldots, 4$, are particular cases of the Kampé de Fériet function, namely

$$F_1 = F_{1:0;0}^{1:1;1}, \quad F_2 = F_{0:1;1}^{1:1;1}, \quad F_3 = F_{1:0;0}^{0:2;2}, \quad F_4 = F_{0:1;1}^{2:0;0}. \tag{3}$$

See [6] (Equations (2a)–(2d)). These functions possess integral representations [6] (Equations (18)–(22)) generalizing the classical Euler integral

$${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\middle| z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}dt}{(1-tz)^a}, \tag{4}$$

convergent for $\Re(b) > 0$, $\Re(c - b) > 0$ and $z \in \mathbb{C} \setminus [1, \infty)$. A natural extension of (4) to the function ${}_{p+1}F_p$ is given by [20] (Equation (2))

$$ {}_{p+1}F_p\left(\left.\begin{matrix}\sigma,\mathbf{a}\\ \mathbf{b}\end{matrix}\right|z\right)=\frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}\int_0^1\frac{1}{(1-tz)^\sigma}G^{p,0}_{p,p}\left(t\left|\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right.\right)\frac{dt}{t}, \tag{5}$$

where $G^{p,0}_{p,p}$ is Meijer–Nørlund G function whose detailed definition and properties can be found in [21] (Section 1). Formula (5) is valid for $z\in\mathbb{C}\setminus[1,\infty)$ if $\Re a_i>0$ for $i=1,\dots,p$ and $\Re\sum_{i=1}^p(b_i-a_i)>0$, while σ is an arbitrary complex number. Representation (5) proved to be very useful for demonstrating and discovering various known and new properties of the function ${}_{p+1}F_p$, as we have shown in a series of papers; see [21] and the references therein. Note that representation (5) is an immediate consequence of the Mellin transform formula for the Meijer–Nørlund G function [21] (Equation (1.4)), namely,

$$ \frac{(\mathbf{a})_n}{(\mathbf{b})_n}=\frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}\int_0^1 t^n G^{p,0}_{p,p}\left(t\left|\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right.\right)\frac{dt}{t},\quad n=0,1,2,\dots \tag{6}$$

valid for $\Re(\mathbf{a})>0$, $\Re\sum_{i=1}^p(b_i-a_i)>0$. Similarly, term-wise integration (assuming its legitimacy) yields the following relations for Kampé de Fériet function ($\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{C}^v$):

$$ F^{p+v:k;l}_{q+v:r;s}\left[\begin{matrix}(\mathbf{a},\boldsymbol{\alpha}):\mathbf{b};\mathbf{c}\\ (\mathbf{d},\boldsymbol{\beta}):\mathbf{e};\mathbf{f}\end{matrix};x,y\right]=\frac{\Gamma(\boldsymbol{\beta})}{\Gamma(\boldsymbol{\alpha})}\int_0^1 G^{v,0}_{v,v}\left(t\left|\begin{matrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{matrix}\right.\right)F^{p:k;l}_{q:r;s}\left[\begin{matrix}\mathbf{a}:\mathbf{b};\mathbf{c}\\ \mathbf{d}:\mathbf{e};\mathbf{f}\end{matrix};xt,yt\right]\frac{dt}{t}, \tag{7}$$

$$ F^{p:k+v;l}_{q:r+v;s}\left[\begin{matrix}\mathbf{a}:(\mathbf{b},\boldsymbol{\alpha});\mathbf{c}\\ \mathbf{d}:(\mathbf{e},\boldsymbol{\beta});\mathbf{f}\end{matrix};x,y\right]=\frac{\Gamma(\boldsymbol{\beta})}{\Gamma(\boldsymbol{\alpha})}\int_0^1 G^{v,0}_{v,v}\left(t\left|\begin{matrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{matrix}\right.\right)F^{p:k;l}_{q:r;s}\left[\begin{matrix}\mathbf{a}:\mathbf{b};\mathbf{c}\\ \mathbf{d}:\mathbf{e};\mathbf{f}\end{matrix};xt,y\right]\frac{dt}{t}, \tag{8}$$

$$ F^{p:k;l+v}_{q:r;s+v}\left[\begin{matrix}\mathbf{a}:\mathbf{b};(\mathbf{c},\boldsymbol{\alpha})\\ \mathbf{d}:\mathbf{e};(\mathbf{f},\boldsymbol{\beta})\end{matrix};x,y\right]=\frac{\Gamma(\boldsymbol{\beta})}{\Gamma(\boldsymbol{\alpha})}\int_0^1 G^{v,0}_{v,v}\left(t\left|\begin{matrix}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{matrix}\right.\right)F^{p:k;l}_{q:r;s}\left[\begin{matrix}\mathbf{a}:\mathbf{b};\mathbf{c}\\ \mathbf{d}:\mathbf{e};\mathbf{f}\end{matrix};x,yt\right]\frac{dt}{t}. \tag{9}$$

In other words, we can extend any pair of parameter vectors by appropriate integration with respect to the Meijer–Nørlund G density. Extending the top/bottom parameters by vectors of different sizes is also possible by using the suitable $G^{p,0}_{q,p}$ functions, but we will not need it in this note. Closely related representations can be found in [22]. Particular cases of the above formulas with beta density

$$ G^{1,0}_{1,1}\left(t\left|\begin{matrix}\beta-1\\ \alpha-1\end{matrix}\right.\right)=\frac{t^{\alpha-1}(1-t)_+^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} \tag{10}$$

were explored in [23]. Formulas (7)–(9) were obtained by formal term-wise integration which should be justified for each specific set of parameter values individually. For the parameters considered below, this can be achieved by employing the Lebesgue dominated convergence theorem in the form given in [24] (Theorem 3.6.2) and using the asymptotic behavior of Meijer–Nørlund's G function in the neighborhoods of the singular points 0 and 1; see [20] (Equation (10) and (11)).This procedure is explained in detail in the proof of Lemma 1 in our paper [25]. An analogous reasoning can be used in the propositions presented below. Given this analogy, we will not present further justifications in our proofs below.

By taking $p=q=0$ in (7) and employing (2) after changing notation, we arrive at

Proposition 1. *Suppose $p\geq 1$, $k\geq 0$, $l\geq 0$, $\Re(\mathbf{a})>0$ and $\Re\sum_{i=1}^p(d_i-a_i)>0$. Then for all $x,y\in\mathbb{C}\setminus[1,\infty)$ we have*

$$F^{p:k+1:l+1}_{p:k:l}\left[\begin{array}{c}\mathbf{a}:\mathbf{b};\mathbf{c}\\ \mathbf{d}:\mathbf{e};\mathbf{f}\end{array};x,y\right]=\frac{\Gamma(\mathbf{d})}{\Gamma(\mathbf{a})}\int_0^1 G^{p,0}_{p,p}\left(t\left|\begin{array}{c}\mathbf{d}\\ \mathbf{a}\end{array}\right.\right){}_{k+1}F_k\left(\begin{array}{c}\mathbf{b}\\ \mathbf{e}\end{array}\middle|xt\right){}_{l+1}F_l\left(\begin{array}{c}\mathbf{c}\\ \mathbf{f}\end{array}\middle|yt\right)\frac{dt}{t}. \tag{11}$$

If further $k,l\geq 1$ *and* $\Re\mathbf{b}_{[1]},\Re\mathbf{c}_{[1]}>0$, $\Re\sum_{i=1}^{k}(e_i-b_{i+1})>0$, $\Re\sum_{i=1}^{l}(f_i-c_{i+1})>0$, *then*

$$F^{p:k+1:l+1}_{p:k:l}\left[\begin{array}{c}\mathbf{a}:\mathbf{b};\mathbf{c}\\ \mathbf{d}:\mathbf{e};\mathbf{f}\end{array};x,y\right]=\frac{\Gamma(\mathbf{d})\Gamma(\mathbf{e})\Gamma(\mathbf{f})}{\Gamma(\mathbf{b}_{[1]})\Gamma(\mathbf{c}_{[1]})\Gamma(\mathbf{a})}$$
$$\times\int_{[0,1]^3}G^{p,0}_{p,p}\left(t\left|\begin{array}{c}\mathbf{d}\\ \mathbf{a}\end{array}\right.\right)G^{k,0}_{k,k}\left(v\left|\begin{array}{c}\mathbf{e}\\ \mathbf{b}_{[1]}\end{array}\right.\right)G^{l,0}_{l,l}\left(u\left|\begin{array}{c}\mathbf{f}\\ \mathbf{c}_{[1]}\end{array}\right.\right)\frac{dudvdt}{uvt(1-xtv)^{b_1}(1-ytu)^{c_1}}. \tag{12}$$

The convergence conditions in this proposition follow from the asymptotic approximations for $G^{p,0}_{p,p}(t)$ near $t=0$ and $t=1$; see [20] (Equations (10) and (11)). Comparing these conditions with those given below Formula (1) (case (ii)), we see that Formula (11) yields analytic continuation of this type of Kampé de Fériet function to $(\mathbb{C}\setminus[1,\infty))^2$ under the restrictions on parameters stated in the proposition. Representation (11) generalizes the Euler-type integral for Appell's function F_1 due to Picard [6] (Equation (22)). Indeed, setting $p=1$, $k=0$, $l=0$ and employing (10), we recover [6] (Equation (22)); see also [3] (Equation 5.8 (5)) and [7] (Theorem 3.41). Another integral representation we will exploit here generalizes the Euler-type integral for the Appell function F_2; see [6] (Equation (19)), [3] (Equation 5.8 (2)) or [7] (Theorem 3.41).

Proposition 2. *Suppose the following conditions on parameters are satisfied:*

(i) $\mathbf{a},\mathbf{b}\in\mathbb{C}^p$, $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{C}^q$; $\boldsymbol{\gamma},\boldsymbol{\lambda}\in\mathbb{C}^k$,

(ii) $\Re(\mathbf{a}),\Re(\boldsymbol{\alpha}),\Re(\boldsymbol{\lambda})>0$, $\Re\sum_{i=1}^{p}(b_i-a_i)>0$, $\Re\sum_{i=1}^{q}(\beta_i-\alpha_i)>0$, $\Re\sum_{i=1}^{k}(\gamma_i-\lambda_i)>0$.

Then for all $x,y\in\mathbb{C}\setminus[1,\infty)$, *we have*

$$F^{p+1:q;k}_{p:q;k}\left[\begin{array}{c}\sigma,\mathbf{a}:\boldsymbol{\alpha};\boldsymbol{\lambda}\\ \mathbf{b}:\boldsymbol{\beta};\boldsymbol{\gamma}\end{array};x,y\right]=\frac{\Gamma(\mathbf{b})\Gamma(\boldsymbol{\beta})\Gamma(\boldsymbol{\gamma})}{\Gamma(\mathbf{a})\Gamma(\boldsymbol{\alpha})\Gamma(\boldsymbol{\lambda})}$$
$$\times\int_{[0,1]^3}G^{p,0}_{p,p}\left(t\left|\begin{array}{c}\mathbf{b}\\ \mathbf{a}\end{array}\right.\right)G^{q,0}_{q,q}\left(u\left|\begin{array}{c}\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{array}\right.\right)G^{k,0}_{k,k}\left(v\left|\begin{array}{c}\boldsymbol{\gamma}\\ \boldsymbol{\lambda}\end{array}\right.\right)\frac{dudvdt}{uvt(1-uxt-vyt)^{\sigma}}. \tag{13}$$

In accordance with (3), the representation for the Appell function F_2 is obtained by setting $p=0$ and $q=k=1$ in view of (10). The proof of Proposition 2 requires a small calculation and will be given in Section 2 below.

Our third proposition generalizes the integral representation for F_3; see [6] (Equation (20)), [3] (Equation 5.8 (3)) or [7] (Theorem 3.41).

Proposition 3. *Suppose the following conditions on parameters are satisfied:*

(i) $\mathbf{a},\mathbf{b}\in\mathbb{C}^p$, $\boldsymbol{\lambda},\boldsymbol{\gamma}\in\mathbb{C}^k$, $\boldsymbol{\alpha}\in\mathbb{C}^q$, $\boldsymbol{\beta}\in\mathbb{C}^{q-1}$;

(ii) $\Re(\mathbf{a}),\Re(\boldsymbol{\lambda}),\Re(\boldsymbol{\alpha}),\Re(g),\Re(c-g)>0$;

(iii) $\Re\sum_{i=1}^{p}(b_i-a_i),\Re\sum_{i=1}^{k}(\gamma_i-\lambda_i)>0$, $\Re\sum_{i=1}^{q-1}(\beta_i-\alpha_i)>\Re(\alpha_q+g-c)$.

Then, for all $x, y \in \mathbb{C} \setminus [1, \infty)$ *we have*

$$F^{p:k+2;q+1}_{p+1:k;q-1}\left[\begin{matrix}\mathbf{a}: & e, g, \boldsymbol{\lambda}; & \delta, \boldsymbol{\alpha} \\ \mathbf{b}, c: & \gamma; & \boldsymbol{\beta}\end{matrix}; x, y\right] = \frac{\Gamma(\mathbf{b})\Gamma(c)\Gamma(\gamma)\Gamma(\boldsymbol{\beta})}{\Gamma(\mathbf{a})\Gamma(g)\Gamma(\boldsymbol{\lambda})\Gamma(\boldsymbol{\alpha})} \int_{[0,1]^4} \frac{v^g(1-v)^{c-g-1}\rho(t,w,u)dV}{(1-xtwv)^e(1-yt(1-v)u)^\delta uvwt}, \quad (14)$$

where $dV = dudvdwdt$ *and*

$$\rho(t,w,u) = G^{p,0}_{p,p}\left(t\left|\begin{matrix}\mathbf{b} \\ \mathbf{a}\end{matrix}\right.\right) G^{k,0}_{k,k}\left(w\left|\begin{matrix}\gamma \\ \boldsymbol{\lambda}\end{matrix}\right.\right) G^{q,0}_{q,q}\left(u\left|\begin{matrix}c-g, \boldsymbol{\beta} \\ \boldsymbol{\alpha}\end{matrix}\right.\right).$$

In accordance with (3), the representation for the Appell function F_3 is obtained by setting $p = k = 0$ and $q = 1$ in (14). Propositions 2 and 3 can also be viewed as formulas of analytic continuation of the corresponding Kampé de Fériet functions to the domain $(\mathbb{C} \setminus [1, \infty))^2$. The situation with the function F_4 is more complicated and its examination is postponed to a future publication.

The main goal of this note is to present two applications of the integral representations (11)–(14). In the subsequent Section 2, we will explore conditions ensuring complete monotonicity of the Kampé de Fériet function which entails log-convexity and other inequalities. In Section 3, we show how known reduction formulas for Appell's functions can be extended to the Kampé de Fériet function. We further show that product formulas for hypergeometric functions of one variable lead to reduction formulas for certain Kampé de Fériet functions and give examples that include some recently discovered product formulas. In the concluding Section 4, we again exploit the integration against Meijer–Nørlund density combined with series rearrangement techniques to derive certain double hypergeometric series transformations which can also be written in terms of Kampé de Fériet functions. Note that the method of integration against Meijer–Nørlund density was systematically applied by us in [26] to reprove many known and discover some new identities for the generalized hypergeometric function of one variable.

2. Some Completely Monotonic Kampé de Fériet Functions

Recall that a real-valued function $f : \mathbb{R}^2_{>0} \to \mathbb{R}$ on the positive orthant is called completely monotonic if [27] (Equation (5.1))

$$(-1)^{m+n}\frac{\partial^{m+n} f(x,y)}{\partial x^m \partial y^n} \geq 0, \quad m, n = 0, 1, 2, \ldots \quad (15)$$

for all $(x, y) \in \mathbb{R}^2_{>0}$. Univariate complete monotonicity plays an important role in studying special functions; see, for instance, [28]. Extending the classical univariate Bernstein's theorem, Schoenberg [27] (Theorem 5.1) showed that these functions are (bivariate) Laplace transforms of non-negative measures,

$$f(x,y) = \int_{\mathbb{R}^2_{>0}} e^{-xt-ys} d\mu(t,s),$$

where the measure is assumed to render the integral finite for all $x, y > 0$. This result was later extended to general cones; see [29]. Complete monotonicity implies that f is jointly log-convex on $\mathbb{R}^2_{>0}$; see [30] (Example 3.41). This means by definition that for any two points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2_{>0}$ and any $0 \leq \lambda \leq 1$, we have [30] (Section 3.5.1)

$$f(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \leq f(x_1, y_1)^\lambda f(x_2, y_2)^{1-\lambda}.$$

In particular, the value of a log-convex function at the average of two points is at most the geometric mean of the values at those points. It is also clear that a log-convex function is quasiconvex, since the logarithm is monotone increasing. If a log-convex function f is twice continuously differentiable, then the Hessian matrix $\nabla^2(\log f)(x,y)$ is positive semi-definite [30] (Section 3.5.2), which implies the following inequalities:

$$ff_{xx} \geq f_x^2, \quad ff_{yy} \geq f_y^2,$$

and

$$\left(ff_{xx} - f_x^2\right)\left(ff_{yy} - f_y^2\right) \geq \left(ff_{xy} - f_x f_y\right)^2.$$

In this section, we will provide sufficient conditions for the Kampé de Fériet functions considered above to be completely monotonic and hence jointly log-convex, satisfying the above inequalities. As our integral representations (11)–(14) contain the Meijer–Nørlund function $G_{p,p}^{p,0}$, we will need conditions that ensure the non-negativity and integrability of this function on the interval of integration. We explored such conditions in our paper [31]. In order to formulate them we need the following definition.

Definition 1. *We will say that a pair of real p-tuples* $\mathbf{a} = (a_1, \ldots, a_p)$, $\mathbf{b} = (b_1, \ldots, b_p)$ *satisfies condition* $\mathrm{PI}(\mathbf{a}, \mathbf{b})$ *if*

$$\begin{aligned} &(1) \quad v(t; \mathbf{a}, \mathbf{b}) = \textstyle\sum_{i=1}^{p} (t^{a_i} - t^{b_i}) > 0 \text{ for } 0 < t < 1; \\ &(2) \quad a_i > 0, i = 1, \ldots, p; \\ &(3) \quad \textstyle\sum_{i=1}^{p} (b_i - a_i) > 0. \end{aligned} \tag{16}$$

As shown in [31] (p. 140), the condition $\mathrm{PI}(\mathbf{a}, \mathbf{b})$ *guarantees non-negativity and integrability of the function*

$$t \to G_{p,p}^{p,0}\left(t \,\middle|\, \begin{matrix} \mathbf{b} \\ \mathbf{a} \end{matrix}\right)$$

on the interval $0 < t < 1$. *In particular, the integral (6) is finite for all non-negative n. Note that the condition* $v(t; \mathbf{a}, \mathbf{b}) > 0$ *in (16) follows from a slightly stronger condition known as weak majorization, which is given by [31] (p. 138)*

$$0 < a_1 \leq a_2 \leq \cdots \leq a_p, \;\; 0 < b_1 \leq b_2 \leq \cdots \leq b_p, \; \sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i, \; k = 1, 2 \ldots, p.$$

With these preparations, we are ready to formulate.

Theorem 1. *Suppose* $\mathbf{a}, \mathbf{d} \in \mathbb{R}^p$, $\boldsymbol{\lambda}, \mathbf{e} \in \mathbb{R}^k$, $\boldsymbol{\alpha}, \mathbf{f} \in \mathbb{R}^l$, *and* $\sigma, \delta > 0$. *If the parameters satisfy the conditions* $\mathrm{PI}(\mathbf{a}, \mathbf{d})$, $\mathrm{PI}(\boldsymbol{\lambda}, \mathbf{e})$, $\mathrm{PI}(\boldsymbol{\alpha}, \mathbf{f})$, *then the function*

$$(x, y) \to F_{p:k:l}^{p:k+1:l+1}\left[\begin{matrix} \mathbf{a}: & \sigma, \boldsymbol{\lambda}; & \delta, \boldsymbol{\alpha} \\ \mathbf{d}: & \mathbf{e}; & \mathbf{f} \end{matrix}; -x, -y\right]$$

is completely monotonic. If $k = 0$ *and/or* $l = 0$, *the corresponding vectors are understood as empty.*

Proof of Theorem 1. The proof consists in application of definition (15) to the integral representation (12). Indeed, the function

$$(x,y) \to (1+ax)^{-b}(1+cy)^{-d},$$

is completely monotonic for any fixed $a,b,c,d > 0$. Hence, differentiating under the integral sign and using non-negativity of all G functions in (12), we immediately arrive at the conclusion. It remains to note that if $k,l \geq 1$, then (12) follows from (11) by substituting the integral representations (5) for the hypergeometric functions ${}_{k+1}F_k$ and ${}_{l+1}F_l$ into (11). If $k=0$ and/or $l=0$, we use the binomial theorem ${}_1F_0(a;-;x) = (1-x)^{-a}$ for the missing factor(s). In turn, Formula (11) was already established before Proposition 1. □

Theorem 2. *Suppose* $\mathbf{a},\mathbf{b} \in \mathbb{R}^p$, $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{R}^q$. $\boldsymbol{\lambda},\boldsymbol{\gamma} \in \mathbb{R}^k$ *and* $\sigma > 0$. *Then, if the parameters satisfy the conditions* $\mathrm{PI}(\mathbf{a},\mathbf{b})$, $\mathrm{PI}(\boldsymbol{\alpha},\boldsymbol{\beta})$, $\mathrm{PI}(\boldsymbol{\lambda},\boldsymbol{\gamma})$, *then the function*

$$(x,y) \to F^{p+1:q;k}_{p:q;k}\left[\begin{matrix}\sigma, \mathbf{a}:\boldsymbol{\alpha};\boldsymbol{\lambda}\\ \mathbf{b}:\boldsymbol{\beta};\boldsymbol{\gamma}\end{matrix};-x,-y\right]$$

is completely monotonic.

Proof of Theorem 2. The proof follows from representation (13) in view of the complete monotonicity of the kernel

$$(x,y) \to (1+utx+vty)^{-\sigma}$$

for any $u,v,t,\sigma > 0$, which can be verified directly, and the non-negativity of the G function weights. It remains to prove Proposition 2. □

Proof of Proposition 2. We have

$$\frac{1}{(1-xu-yv)^{\sigma}} = \sum_{n=0}^{\infty}\frac{(\sigma)_n v^n y^n}{n!(1-xu)^{\sigma+n}} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\sigma)_n(\sigma+n)_m v^n y^n u^m x^m}{n!m!}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\sigma)_{n+m} v^n y^n u^m x^m}{n!m!} = F^{1:0:0}_{0:0:0}\left[\begin{matrix}\sigma:-:-\\ -:-:-\end{matrix};ux,vy\right]. \quad (17)$$

Suppose $\mathbf{a},\mathbf{b} \in \mathbb{C}^p$, $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{C}^q$, $\boldsymbol{\gamma},\boldsymbol{\lambda} \in \mathbb{C}^k$, $\Re\mathbf{a}, \Re(\boldsymbol{\alpha}), \Re(\boldsymbol{\lambda}) > 0$, $\Re\sum_{i=1}^{p}(b_i - a_i) > 0$, $\Re\sum_{i=1}^{q}(\beta_i - \alpha_i) > 0$, $\Re\sum_{i=1}^{k}(\gamma_i - \lambda_i) > 0$. Multiplying (17) by the appropriate G functions and integrating term-wise, we obtain (13), which is then extended by analytic continuation to all $x,y \in \mathbb{C}\setminus[1,\infty)$. □

Theorem 3. *Suppose* $\mathbf{a},\mathbf{b} \in \mathbb{R}^p$, $\boldsymbol{\lambda},\boldsymbol{\gamma} \in \mathbb{R}^k$, $\boldsymbol{\alpha} \in \mathbb{R}^q$, $\boldsymbol{\beta} \in \mathbb{R}^{q-1}$, *and* $g,e,\delta > 0$, $c > g$. *If the parameters satisfy the conditions* $\mathrm{PI}(\mathbf{a},\mathbf{b})$, $\mathrm{PI}(\boldsymbol{\lambda},\boldsymbol{\gamma})$, $\mathrm{PI}(\boldsymbol{\alpha},(c-g,\boldsymbol{\beta}))$, *then the function*

$$(x,y) \to F^{p:k+2;q+1}_{p+1:k;q-1}\left[\begin{matrix}\mathbf{a}: & e,g,\boldsymbol{\lambda}; & \delta,\boldsymbol{\alpha}\\ \mathbf{b},c: & \boldsymbol{\gamma}; & \boldsymbol{\beta}\end{matrix};-x,-y\right]$$

is completely monotonic.

Proof of Theorem 3. The proof follows from representation (14) by the argument given in the proof of Theorem 1. Finally, let us prove Proposition 3. □

Proof of Proposition 3. For sufficiently small (x,y), we have

$$\begin{aligned}
&\int_{[0,1]^3} G_{p,p}^{p,0}\left(t\left|\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right.\right)\frac{v^g(1-v)^{c-g-1}}{(1-xtv)^e(1-yt(1-v)u)^\delta}G_{q,q}^{q,0}\left(u\left|\begin{matrix}c-g,\boldsymbol{\beta}\\ \boldsymbol{\alpha}\end{matrix}\right.\right)\frac{dudvdt}{uvt}\\
&\quad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{[0,1]^3} G_{p,p}^{p,0}\left(t\left|\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right.\right)\frac{(e)_m(\delta)_n x^m y^n v^{m+g-1}(1-v)^{n+c-g-1}u^n t^{n+m}}{n!m!}\\
&\qquad\qquad\times G_{q,q}^{q,0}\left(u\left|\begin{matrix}c-g-1,\boldsymbol{\beta}-1\\ \boldsymbol{\alpha}-1\end{matrix}\right.\right)dudvdt\\
&\quad=\frac{\Gamma(\mathbf{a})\Gamma(\boldsymbol{\alpha})}{\Gamma(\mathbf{b})\Gamma(\boldsymbol{\beta})}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(e)_m(\delta)_n(\mathbf{a})_{n+m}(\boldsymbol{\alpha})_n}{(\mathbf{b})_{n+m}\Gamma(c-g+n)(\boldsymbol{\beta})_n}\frac{\Gamma(g+m)\Gamma(c-g+n)}{\Gamma(c+m+n)}\frac{x^m y^n}{m!n!}\\
&\qquad=\frac{\Gamma(\mathbf{a})\Gamma(\boldsymbol{\alpha})\Gamma(g)}{\Gamma(\mathbf{b})\Gamma(\boldsymbol{\beta})\Gamma(c)}F_{p+1:0;q-1}^{p:2;q+1}\left[\begin{matrix}\mathbf{a}: & e,g; & \delta,\boldsymbol{\alpha}\\ \mathbf{b},c: & -; & \boldsymbol{\beta}\end{matrix};x,y\right].
\end{aligned}$$

Then application of (8) leads to (14). The result is then extended to all $x,y\in\mathbb{C}\setminus[1,\infty)$ by analytic continuation. □

3. Reduction Formulas for Kampé de Fériet Function

We will assume in this section that $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{C}^v$. Formulas (7)–(9) provide a simple method for concatenating additional parameters to known transformation formulas by using multiplication by $G_{v,v}^{v,0}$ and subsequent integration. More concretely, assuming that for any $t\in(0,1)$ there holds

$$F_{q:r;s}^{p:k;l}\left[\begin{matrix}\mathbf{a}:\mathbf{b};\mathbf{c}\\ \mathbf{d}:\mathbf{e};\mathbf{f}\end{matrix};xt,yt\right]=F_{q_1:r_1;s_1}^{p_1:k_1;l_1}\left[\begin{matrix}\mathbf{a}_1:\mathbf{b}_1;\mathbf{c}_1\\ \mathbf{d}_1:\mathbf{e}_1;\mathbf{f}_1\end{matrix};x_1t,y_1t\right],\tag{18}$$

where x_1,y_1 are certain functions of x,y, we can apply (7) on both sides of (18) to obtain

$$F_{q+v:r;s}^{p+v:k;l}\left[\begin{matrix}(\mathbf{a},\boldsymbol{\alpha}):\mathbf{b};\mathbf{c}\\ (\mathbf{d},\boldsymbol{\beta}):\mathbf{e};\mathbf{f}\end{matrix};x,y\right]=F_{q_1+v:r_1;s_1}^{p_1+v:k_1;l_1}\left[\begin{matrix}(\mathbf{a}_1,\boldsymbol{\alpha}):\mathbf{b}_1;\mathbf{c}_1\\ (\mathbf{d}_1,\boldsymbol{\beta}):\mathbf{e}_1;\mathbf{f}_1\end{matrix};x_1,y_1\right].\tag{19}$$

Similarly, if any $t\in(0,1)$

$$F_{q:r;s}^{p:k;l}\left[\begin{matrix}\mathbf{a}:\mathbf{b};\mathbf{c}\\ \mathbf{d}:\mathbf{e};\mathbf{f}\end{matrix};xt,y\right]=F_{q_1:r_1;s_1}^{p_1:k_1;l_1}\left[\begin{matrix}\mathbf{a}_1:\mathbf{b}_1;\mathbf{c}_1\\ \mathbf{d}_1:\mathbf{e}_1;\mathbf{f}_1\end{matrix};x_1t,y_1\right],\tag{20}$$

then by (8) we obtain

$$F_{q:r+v;s}^{p:k+v;l}\left[\begin{matrix}\mathbf{a}:(\mathbf{b},\boldsymbol{\alpha});\mathbf{c}\\ \mathbf{d}:(\mathbf{e},\boldsymbol{\beta});\mathbf{f}\end{matrix};x,y\right]=F_{q_1+v:r_1;s_1}^{p_1+v:k_1;l_1}\left[\begin{matrix}\mathbf{a}_1:(\mathbf{b}_1,\boldsymbol{\alpha});\mathbf{c}_1\\ \mathbf{d}_1:(\mathbf{e}_1,\boldsymbol{\beta});\mathbf{f}_1\end{matrix};x_1,y_1\right].\tag{21}$$

A similar trick can be applied to certain reduction formulas. Namely, if

$$F_{q:r;s}^{p:k;l}\left[\begin{matrix}\mathbf{a}:\mathbf{b};\mathbf{c}\\ \mathbf{d}:\mathbf{e};\mathbf{f}\end{matrix};xt,yt\right]={}_{p_1}F_{q_1}\left(\left.\begin{matrix}\mathbf{a}_1\\ \mathbf{b}_1\end{matrix}\right|zt\right),\tag{22}$$

where $z=z(x,y)$, then

$$F_{q+v:r;s}^{p+v:k;l}\left[\begin{matrix}(\mathbf{a},\boldsymbol{\alpha}):\mathbf{b};\mathbf{c}\\ (\mathbf{d},\boldsymbol{\beta}):\mathbf{e};\mathbf{f}\end{matrix};x,y\right]={}_{p_1+v}F_{q_1+v}\left(\left.\begin{matrix}\mathbf{a}_1,\boldsymbol{\alpha}\\ \mathbf{b}_1,\boldsymbol{\beta}\end{matrix}\right|z\right).\tag{23}$$

Below we exploit this simple idea combined with Propositions 1–3 to derive some reduction and transformations formulas, most of which are believed to be new. Via analytic continuation, all the reduction formulas below hold whenever the Kampé de Fériet functions on both sides are well defined.

Example 1. *Rewrite the formula [19] (Equation (16.16.2))*

$$F_2(\alpha;\beta,\beta';\gamma,\beta';x,y)=(1-y)^{-\alpha}\,{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\middle|\,\frac{x}{1-y}\right), \tag{24}$$

where F_2 is Appell's function, in terms of the Kampé de Fériet function in view of (3) and (2):

$$F^{1:1;0}_{0:1;0}\left[\begin{matrix}\alpha:\beta;-\\ -:\gamma;-\end{matrix};xt,y\right]={}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\middle|\,\frac{xt}{1-y}\right){}_1F_0\left(\begin{matrix}\alpha\\ -\end{matrix}\middle|\,y\right)$$
$$=F^{0:2;1}_{0:1;0}\left[\begin{matrix}-:&\alpha,\beta;&\alpha\\ -:&\gamma;&-\end{matrix};\frac{xt}{1-y},y\right].$$

Then, applying (20)–(21), we obtain a presumably new reduction formula:

$$F^{1:1+v;0}_{0:1+v;0}\left[\begin{matrix}\alpha:(\beta,\boldsymbol{\alpha});-\\ -:(\gamma,\boldsymbol{\beta});-\end{matrix};x,y\right]=(1-y)^{-\alpha}\,{}_{v+2}F_{v+1}\left(\begin{matrix}\alpha,\beta,\boldsymbol{\alpha}\\ \gamma,\boldsymbol{\beta}\end{matrix}\middle|\,\frac{x}{1-y}\right).$$

Example 2. *According to [19] (Equation (16.16.3)),*

$$F_2(\alpha;\beta,\beta';\gamma,\alpha;x,y)=(1-y)^{-\beta'}F_1\left(\beta;\alpha-\beta',\beta';\gamma;x,\frac{x}{1-y}\right), \tag{25}$$

where F_1, F_2 are Appell's functions. Then in view of (3) we can write

$$F^{1:1;1}_{0:1;1}\left[\begin{matrix}\alpha:\beta;\beta'\\ -:\gamma;\alpha\end{matrix};xt,y\right]=(1-y)^{-\beta'}F^{1:1;1}_{1:0;0}\left[\begin{matrix}\beta:&\alpha-\beta';&\beta'\\ \gamma:&-;&-\end{matrix};xt,\frac{xt}{1-y}\right].$$

Hence, applying (18)–(19), we obtain a presumably new reduction formula:

$$F^{1:v+1;1}_{0:v+1;1}\left[\begin{matrix}\alpha:(\beta,\boldsymbol{\alpha});\beta'\\ -:(\gamma,\boldsymbol{\beta});\alpha\end{matrix};x,y\right]=(1-y)^{-\beta'}F^{v+1:1;1}_{v+1:0;0}\left[\begin{matrix}(\beta,\boldsymbol{\alpha}):&\alpha-\beta';&\beta'\\ (\gamma,\boldsymbol{\beta}):&-;&-\end{matrix};x,\frac{x}{1-y}\right]. \tag{26}$$

Example 3. *The celebrated Clausen's product identity reads [18] (pp. 116, 180)*

$$\left[{}_2F_1\left(\begin{matrix}c,d\\ c+d+1/2\end{matrix}\middle|\,x\right)\right]^2={}_3F_2\left(\begin{matrix}2c,2d,c+d\\ c+d+1/2,2c+2d\end{matrix}\middle|\,x\right).$$

Then, writing the left hand side as a Kampé de Fériet function by (2) and applying (22)–(23) yields

$$F^{v:2;2}_{v:1;1}\left[\begin{matrix}\boldsymbol{\alpha}:&(c,d);&(c,d)\\ \boldsymbol{\beta}:&c+d+1/2;&c+d+1/2\end{matrix};x,x\right]={}_{v+3}F_{v+2}\left(\begin{matrix}2c,2d,c+d,\boldsymbol{\alpha}\\ c+d+1/2,2c+2d,\boldsymbol{\beta}\end{matrix}\middle|\,x\right).$$

We could not locate the above identity in the existing literature.

Example 4. *Askey found an extension of Clausen's identity given by [32] (Equation (3.7)):*

$$\left[{}_2F_1\left(\begin{matrix}a,b\\ a+b+\frac{3}{2}\end{matrix}\,\middle|\,x\right)\right]^2={}_5F_4\left(\begin{matrix}2a,2b,a+b,\zeta_1+1,\zeta_2+1\\ a+b+\frac{3}{2},2a+2b+2,\zeta_1,\zeta_2\end{matrix}\,\middle|\,x\right),$$

where ζ_1,ζ_2 are negated zeros of the second-degree polynomial

$$t\to t^2+(4a+4b+1+8ab)t+2(a+b)(2a+1)(2b+1).$$

Then, writing the left hand side as a Kampé de Fériet function by (2) and applying (22)–(23), we obtain a new formula:

$$F^{v:2;2}_{v:1;1}\left[\begin{matrix}\boldsymbol{\alpha}: & (a,b); & (a,b)\\ \boldsymbol{\beta}: & a+b+3/2; & a+b+3/2\end{matrix};x,x\right]$$
$$={}_{v+5}F_{v+4}\left(\begin{matrix}2a,2b,a+b,\zeta_1+1,\zeta_2+1,\boldsymbol{\alpha}\\ a+b+\frac{3}{2},2a+2b+2,\zeta_1,\zeta_2,\boldsymbol{\beta}\end{matrix}\,\middle|\,x\right).$$

Example 5. *Another product formula has been found recently in [33] (Equation (42)). It reads*

$$\left[{}_3F_2\left(\begin{matrix}c,b,f+1\\ c+b+3/2,f\end{matrix}\,\middle|\,x\right)\right]^2={}_6F_5\left(\begin{matrix}2c,2b,c+b,\lambda_1+1,\lambda_2+1,\lambda_3+1\\ c+b+3/2,2c+2b+2,\lambda_1,\lambda_2,\lambda_3\end{matrix}\,\middle|\,x\right),$$

where $\lambda_1,\lambda_2,\lambda_3$ are the roots of the characteristic polynomial

$$P_3(t)=f(f-t)$$
$$\times(t^2-(8cb+4c+4b+1)t+2(2c+1)(2b+1)(c+b))+2cbt(t+1)(c+b-t+1/2).$$

By (2) and (22)–(23), we then obtain the reduction formula:

$$F^{v:3;3}_{v:2;2}\left[\begin{matrix}\boldsymbol{\alpha}: & (c,b,f+1); & (c,b,f+1)\\ \boldsymbol{\beta}: & (c+b+3/2,f); & (c+b+3/2,f)\end{matrix};x,x\right]$$
$$={}_{v+6}F_{v+5}\left(\begin{matrix}2c,2b,c+b,\lambda_1+1,\lambda_2+1,\lambda_3+1,\boldsymbol{\alpha}\\ c+b+3/2,2c+2b+2,\lambda_1,\lambda_2,\lambda_3,\boldsymbol{\beta}\end{matrix}\,\middle|\,x\right).\quad(27)$$

Example 6. *Another product formula, found recently in [34] (Example 1), reads*

$${}_2F_1\left(\begin{matrix}a,b\\ a+b+\frac{1}{2}\end{matrix}\,\middle|\,x\right){}_3F_2\left(\begin{matrix}a,b,f+1\\ a+b+\frac{1}{2},f\end{matrix}\,\middle|\,x\right)={}_4F_3\left(\begin{matrix}2a,2b,a+b,2f+1\\ a+b+\frac{1}{2},2a+2b,2f\end{matrix}\,\middle|\,x\right).$$

This leads to a new reduction formula:

$$F^{v:2;3}_{v:1;2}\left[\begin{matrix}\boldsymbol{\alpha}: & (a,b); & (a,b,f+1)\\ \boldsymbol{\beta}: & a+b+1/2; & (a+b+1/2,f)\end{matrix};x,x\right]$$
$$={}_{v+4}F_{v+3}\left(\begin{matrix}2a,2b,a+b,2f+1,\boldsymbol{\alpha}\\ a+b+\frac{1}{2},2a+2b,2f,\boldsymbol{\beta}\end{matrix}\,\middle|\,x\right).$$

Example 7. *One of the product identities discovered by Orr is (see last formula in [34] (Example 1))*

$$ {}_2F_1\left(\begin{array}{c} a,b \\ a+b+\frac{1}{2} \end{array}\middle| x\right) {}_2F_1\left(\begin{array}{c} a+1,b \\ a+b+\frac{1}{2} \end{array}\middle| x\right) = {}_3F_2\left(\begin{array}{c} 2a+1,2b,a+b \\ a+b+\frac{1}{2},2a+2b \end{array}\middle| x\right). $$

By the previous argument, it implies that

$$ F^{v:2;2}_{v:1;1}\left[\begin{array}{ccc} \boldsymbol{\alpha}: & (a,b); & (a+1,b) \\ \boldsymbol{\beta}: & a+b+1/2; & a+b+1/2 \end{array}; x, x\right] = {}_{v+3}F_{v+2}\left(\begin{array}{c} 2a+1,2b,a+b,\boldsymbol{\alpha} \\ a+b+\frac{1}{2},2a+2b,\boldsymbol{\beta} \end{array}\middle| x\right) $$

which is another identity not found in the existing literature.

4. Further Double Series Transformations

We will use the standard shorthand notation

$$ \Delta(k,a) = \left(\frac{a}{k}, \frac{a+1}{k}, \ldots, \frac{a+k-1}{k}\right) $$

and its extension $\Delta(k,\mathbf{a}) = (\Delta(k,a_1),\ldots,\Delta(k,a_p))$ to a p-tuple $\mathbf{a} = (a_1,\ldots,a_p)$. We will also write $\Delta(k,\mathbf{a})_n$ to denote the product

$$ \Delta(k,\mathbf{a})_n = (a_1/k)_n \cdots ((a_1+k-1)/k)_n \cdots (a_p/k)_n \cdots ((a_p+k-1)/k)_n. $$

We will need the following lemma:

Lemma 1. *Suppose $p \le q$, $\alpha \in \mathbb{C}$, $\mathbf{a} \in \mathbb{C}^p$, $\mathbf{b} \in (\mathbb{C} \setminus -\mathbb{N}_0)^q$ and, when $p = q$, additionally $|x| < 1$, $|y| < 1$. Then,*

$$ \sum_{m\ge 0}\sum_{n=0}^{m} \frac{(\alpha)_m(\mathbf{a})_{m+n}}{(\mathbf{b})_{m+n}} \frac{x^m y^n}{n!(m-n)!} = \sum_{m\ge 0} \frac{(\mathbf{a})_m(\alpha)_m x^m}{(\mathbf{b})_m m!} {}_{2p+1}F_{2q}\left(\begin{array}{c} \Delta(2,\mathbf{a}+m),\alpha+m \\ \Delta(2,\mathbf{b}+m) \end{array}\middle| 4^{p-q}xy\right). \tag{28} $$

Proof of Lemma 1. Begin with Slater's transformation in the following (corrected) form [35] (Section 2):

$$ \sum_{m\ge 0}\sum_{n=0}^{m} \frac{(\mathbf{v})_n(\mathbf{d})_m(\mathbf{u})_{m-n}(\mathbf{a})_{m+n}}{(\mathbf{f})_n(\mathbf{g})_m(\mathbf{e})_{m-n}(\mathbf{b})_{m+n}} \frac{x^m y^n w^{m-n} z^{m+n}}{n!(m-n)!} = \sum_{m,n\ge 0} \frac{(\mathbf{v})_n(\mathbf{d})_{m+n}(\mathbf{u})_m(\mathbf{a})_{m+2n}}{(\mathbf{f})_n(\mathbf{g})_{m+n}(\mathbf{e})_m(\mathbf{b})_{m+2n}} \frac{x^{m+n} y^n w^m z^{m+2n}}{n!m!}. $$

Taking $w = z = 1$, $\mathbf{d} = \alpha$ and other vectors except for $\mathbf{a}$ and $\mathbf{b}$ as empty, we obtain the following particular case:

$$ \sum_{m\ge 0}\sum_{n=0}^{m} \frac{(\alpha)_m(\mathbf{a})_{m+n}}{(\mathbf{b})_{m+n}} \frac{x^m y^n}{n!(m-n)!} = \sum_{m,n\ge 0} \frac{(\alpha)_{m+n}(\mathbf{a})_{m+2n}}{(\mathbf{b})_{m+2n}} \frac{x^m (xy)^n}{m!n!}. $$

On the other hand, according to [14] (Equation (2.13), p. 438) or [15] (Lemma 2), we have

$$ \sum_{m,n\ge 0} \Omega(m+2n)(\alpha)_{m+n} \frac{(z_1+z_2)^m(-z_1z_2)^n}{m!n!} = \sum_{m,n\ge 0} \Omega(m+n)(\alpha)_m(\alpha)_n \frac{z_1^m z_2^n}{m!n!}. $$

We can combine the two formulas by setting

$$ \Omega(k) = \frac{(\mathbf{a})_k}{(\mathbf{b})_k}, \quad x = z_1 + z_2, \quad xy = -z_1z_2. $$

Solving for z_1, z_2 we have

$$z_1 = \frac{x + \sqrt{x^2 + 4xy}}{2}, \quad z_2 = \frac{x - \sqrt{x^2 + 4xy}}{2},$$

so that

$$\sum_{m\geq 0}\sum_{n=0}^{m} \frac{(\alpha)_m(\mathbf{a})_{m+n}}{(\mathbf{b})_{m+n}} \frac{x^m y^n}{n!(m-n)!}$$
$$= \sum_{m,n\geq 0} \frac{(\mathbf{a})_{m+n}}{(\mathbf{b})_{m+n}} (\alpha)_m(\alpha)_n \frac{(x + \sqrt{x^2+4xy})^m (x - \sqrt{x^2+4xy})^n}{2^{m+n} m! n!} = F_{q:0;0}^{p:1;1}\left[\begin{matrix} \mathbf{a}: \alpha; \alpha \\ \mathbf{b}: -; - \end{matrix}; z_1, z_2\right]. \quad (29)$$

Using the expansion [15] (Equation (24)), we then have

$$F_{q:0;0}^{p:1;1}\left[\begin{matrix} \mathbf{a}: \alpha; \alpha \\ \mathbf{b}: -; - \end{matrix}; z_1, z_2\right]$$
$$= \sum_{m\geq 0} \frac{(\mathbf{a})_m(\alpha)_m (z_1+z_2)^m}{(\mathbf{b})_m m!} {}_{2p+1}F_{2q}\left(\begin{matrix} \Delta(2, \mathbf{a}+m), \alpha+m \\ \Delta(2, \mathbf{b}+m) \end{matrix}\middle| -4^{p-q} z_1 z_2\right).$$

Substituting the expressions for z_1, z_2 we obtain (28). □

Theorem 4. *Suppose* $\alpha \in \mathbb{C}$, $\mathbf{a} \in \mathbb{C}^p$, $\mathbf{b} \in (\mathbb{C} \setminus -\mathbb{N}_0)^p$ *and* $|y| < 1$. *Then,*

$$\sum_{m\geq 0} \frac{(\alpha)_m(\mathbf{a})_m (-y)^m}{(\mathbf{b})_m m!} {}_{2p+1}F_{2p}\left(\begin{matrix} \alpha+m, \Delta(2, \mathbf{a}+m) \\ \Delta(2, \mathbf{b}+m) \end{matrix}\middle| -y^2\right)$$
$$= \sum_{m\geq 0} \frac{(-\alpha)_m(\mathbf{a})_m y^m}{(\mathbf{b})_m m!} {}_{3p+1}F_{3p}\left(\begin{matrix} \alpha, \Delta(3, \mathbf{a}+m) \\ \Delta(3, \mathbf{b}+m) \end{matrix}\middle| y^3\right). \quad (30)$$

Proof of Theorem 4. We start with

$$(1+x+x^2)^{-\alpha} = \left(\frac{1-x}{1-x^3}\right)^{\alpha} = \left[\sum_{k=0}^{\infty} \frac{(-\alpha)_k}{k!} x^k\right]\left[\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^{3n}\right] = \sum_{n,k=0}^{\infty} \frac{(-\alpha)_k(\alpha)_n}{k!n!} x^{3n+k}.$$

Set $x = ty$ in the above formula, multiply by $G_{p,p}^{p,0}\left(t\middle|\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right)$ and integrate from zero to one:

$$\int_0^1 \frac{dt}{(1+ty+t^2y^2)^{\alpha}} G_{p,p}^{p,0}\left(t\middle|\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right) = \sum_{n,k=0}^{\infty} \frac{(-\alpha)_k(\alpha)_n}{k!n!} y^{3n+k} \int_0^1 G_{p,p}^{p,0}\left(t\middle|\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right) t^{3n+k} dt$$
$$= \sum_{n,k=0}^{\infty} \frac{(-\alpha)_k(\alpha)_n}{k!n!} y^{3n+k} \frac{\Gamma(\mathbf{a}+3n+k)}{\Gamma(\mathbf{b}+3n+k)} = \sum_{k=0}^{\infty} \frac{(-\alpha)_k \Gamma(\mathbf{a}+k)}{\Gamma(\mathbf{b}+k)k!} y^k \sum_{n=0}^{\infty} \frac{(\alpha)_n \Delta(3,(\mathbf{a}+k))_n}{\Delta(3,(\mathbf{b}+k))_n} \frac{y^{3n}}{n!}$$
$$= \frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{b})} \sum_{k=0}^{\infty} \frac{(-\alpha)_k(\mathbf{a})_k}{(\mathbf{b})_k k!} y^k {}_{3p+1}F_{3p}\left(\begin{matrix} \alpha, \Delta(3, \mathbf{a}+k) \\ \Delta(3, \mathbf{b}+k) \end{matrix}\middle| y^3\right).$$

On the other hand,

$$(1+x+x^2)^{-\alpha} = (1+x(1+x))^{-\alpha} = \sum_{m\geq 0} \frac{(-1)^m(\alpha)_m}{m!} x^m (1+x)^m$$
$$= \sum_{m\geq 0} \frac{(-1)^m(\alpha)_m}{m!} x^m \sum_{k=0}^{m} \binom{m}{k} x^k = \sum_{m,k\geq 0} \binom{-\alpha}{m}\binom{m}{k} x^{m+k}.$$

Again setting $x = ty$ and integrating against G function density, we obtain

$$\int_0^1 \frac{dt}{(1+ty+t^2y^2)^\alpha} G^{p,0}_{p,p}\left(t\left|\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right.\right) = \sum_{m,k\geq 0}\binom{-\alpha}{m}\binom{m}{k} y^{m+k}\int_0^1 t^{m+k} G^{p,0}_{p,p}\left(t\left|\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right.\right)dt$$
$$= \sum_{m,k\geq 0}\binom{-\alpha}{m}\binom{m}{k} y^{m+k}\frac{\Gamma(\mathbf{a}+m+k)}{\Gamma(\mathbf{b}+m+k)} = \frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{b})}\sum_{m\geq 0}\sum_{k=0}^{m}\frac{(\alpha)_m(\mathbf{a})_{m+k}}{(\mathbf{b})_{m+k}}\frac{(-y)^m y^k}{k!(m-k)!}.$$

Finally, applying (28) with $p = q$ and $x = -y$, we arrive at (30). □

Remark 1. *Rearranging both sides of* (30) *as series in powers of y and equating coefficients, we recover the identity underlying* (30)*, namely,*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k(\alpha)_{n-k}}{k!(n-2k)!} = (-1)^n\sum_{k=0}^{\lfloor n/3\rfloor}\frac{(\alpha)_k(-\alpha)_{n-3k}}{k!(n-3k)!}.$$

This identity is probably known, but it is not easy to locate in the literature.

We can obtain a simpler identity by integrating both sides of the relation $1/(1+ty) = (1-ty)/(1-t^2y^2)$ against G function density. Indeed,

$$\int_0^1 \frac{dt}{(1+ty)^\alpha} G^{p,0}_{p,p}\left(t\left|\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right.\right) = \sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}(-y)^k\int_0^1 t^k G^{p,0}_{p,p}\left(t\left|\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right.\right)dt$$
$$= \sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}(-y)^k\frac{\Gamma(\mathbf{a}+k)}{\Gamma(\mathbf{b}+k)} = \frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{b})}{}_{p+1}F_p\left(\begin{matrix}\alpha,\mathbf{a}\\ \mathbf{b}\end{matrix}\,\middle|\,-y\right),$$

and

$$\int_0^1 \frac{(1-ty)^\alpha dt}{(1-t^2y^2)^\alpha} G^{p,0}_{p,p}\left(t\left|\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right.\right) = \sum_{m,k\geq 0}\frac{(-\alpha)_m(\alpha)_k}{k!m!}y^{m+2k}\int_0^1 t^{m+2k} G^{p,0}_{p,p}\left(t\left|\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right.\right)dt$$
$$= \sum_{m,k\geq 0}\frac{(-\alpha)_m(\alpha)_k}{m!k!}y^{m+2k}\frac{\Gamma(\mathbf{a}+m+2k)}{\Gamma(\mathbf{b}+m+2k)} = \sum_{m=0}^{\infty}\frac{(-\alpha)_m\Gamma(\mathbf{a}+m)}{m!\Gamma(\mathbf{b}+m)}y^m$$
$$\times\sum_{k=0}^{\infty}\frac{(\alpha)_k\Delta(2,(\mathbf{a}+m))_k}{\Delta(2,(\mathbf{b}+m))_k}\frac{y^{2k}}{k!} = \sum_{m=0}^{\infty}\frac{(-\alpha)_m\Gamma(\mathbf{a}+m)}{\Gamma(\mathbf{b}+m)m!}y^m\,{}_{2p+1}F_{2p}\left(\begin{matrix}\alpha,\Delta(2,\mathbf{a}+m)\\ \Delta(2,\mathbf{b}+m)\end{matrix}\,\middle|\,y^2\right).$$

Hence, we have

Theorem 5. *Suppose* $\alpha\in\mathbb{C}$, $\mathbf{a}\in\mathbb{C}^p$, $\mathbf{b}\in(\mathbb{C}\setminus -\mathbb{N}_0)^p$ *and* $|y|<1$. *Then,*

$${}_{p+1}F_p\left(\begin{matrix}\alpha,\mathbf{a}\\ \mathbf{b}\end{matrix}\,\middle|\,-y\right) = \sum_{m=0}^{\infty}\frac{(-\alpha)_m(\mathbf{a})_m}{(\mathbf{b})_m m!}y^m\,{}_{2p+1}F_{2p}\left(\begin{matrix}\alpha,\Delta(2,\mathbf{a}+m)\\ \Delta(2,\mathbf{b}+m)\end{matrix}\,\middle|\,y^2\right). \tag{31}$$

Remark 2. *The particular case* $\alpha = 1$ *of* (31) *results in the well-known decomposition into odd and even parts:*

$${}_{p+1}F_p\left(\begin{matrix}1,\mathbf{a}\\ \mathbf{b}\end{matrix}\,\middle|\,-y\right) = {}_{2p+1}F_{2p}\left(\begin{matrix}1,\Delta(2,\mathbf{a})\\ \Delta(2,\mathbf{b})\end{matrix}\,\middle|\,y^2\right) - y\frac{(\mathbf{a})}{(\mathbf{b})}{}_{2p+1}F_{2p}\left(\begin{matrix}1,\Delta(2,\mathbf{a}+1)\\ \Delta(2,\mathbf{b}+1)\end{matrix}\,\middle|\,y^2\right).$$

Note that while the right hand side of (31) resembles the left hand side of (30), it does not seem to be reducible to it.

Now returning to the general setting of Lemma 1, with $\mathbf{a} \in \mathbb{C}^p$ and $\mathbf{b} \in (\mathbb{C} \setminus -\mathbb{N}_0)^q$, we can obtain a related formula from (29). Indeed setting $x = -4y$ in (29), we have

$$\sum_{m\geq 0}\sum_{n=0}^{m}\frac{(\alpha)_m(\mathbf{a})_{m+n}}{(\mathbf{b})_{m+n}}\frac{(-4y)^m y^n}{n!(m-n)!}=\sum_{m,n\geq 0}\frac{(\mathbf{a})_{m+n}}{(\mathbf{b})_{m+n}}(\alpha)_m(\alpha)_n\frac{(-2y)^{m+n}}{m!n!}={}_{p+1}F_q\left(\begin{matrix}\mathbf{a},2\alpha\\ \mathbf{b}\end{matrix}\,\middle|\,-2y\right),$$

since, by [36] (Equation (4.9)),

$$\sum_{m,n\geq 0}\Omega(m+n)(\alpha)_n(\beta)_m\frac{z^{m+n}}{m!n!}=\sum_{n\geq 0}\Omega(n)(\alpha+\beta)_n\frac{z^n}{n!}.$$

Hence, replacing y by $y/2$ we obtain the expansion

$${}_{p+1}F_q\left(\begin{matrix}2\alpha,\mathbf{a}\\ \mathbf{b}\end{matrix}\,\middle|\,-y\right)=\sum_{m\geq 0}\frac{(\mathbf{a})_m(\alpha)_m(-2y)^m}{(\mathbf{b})_m m!}{}_{2p+1}F_{2q}\left(\begin{matrix}\alpha+m,\Delta(2,\mathbf{a}+m)\\ \Delta(2,\mathbf{b}+m)\end{matrix}\,\middle|\,-4^{p-q}y^2\right).$$

This implies that

$$\sum_{m=0}^{j}\frac{(\alpha)_m(-4)^m}{(j-m)!(2m-j)!}=\frac{(-2)^j(2\alpha)_j}{j!}.$$

5. Conclusions and Future Work

In this work we made first steps towards investigations of bivariate hypergeometric functions by using integral representations with G function density. We have considered three types of Kampé de Fériet functions generalizing Appell's functions F_1, F_2, and F_3. For the function F_4, the situation is more complicated and, while it is easy to define its natural generalization in terms of the Kampé de Fériet function, no simple integral representation seems to be available. Those due to Aomoto—see [10] (Equation (0.4)), [7] (Theorem 3.4.1) and Goto-Matsumoto [11] (Equations (2.3)–(3.5))—do not reduce to usual Lebesgue integrals and should be understood in terms of regularization. Horn's list of bivariate hypergeometric series includes a number of series with Pochhammer's symbols containing combinations of summation indices like $(a)_{m-n}$, $(a)_{2m-n}$ and $(a)_{2m+n}$. These types of series can also be generalized by extending their parameter lists. We expect that such generalizations may also possess integral representations, extending the integrals found in [37] for Horn's function H_2 and Olsson's function F_p. We further envision more integral representations involving integration over $(0,\infty)$ for confluent families of Kampé de Fériet functions extending those given in [22].

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